PRODUCTION SYSTEMS ENGINEERING

Chapter 14: Design of Lean Serial Lines with Continuous Models of Machine Reliability

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- Motivation
  - The same as in Bernoulli lines: the need for analytical methods for selecting the smallest buffering, which is necessary and sufficient to ensure the desired efficiency of a production line.
  - The methods developed for Bernoulli lines are generalized here to serial lines with continuous time model of machine reliability.

OUTLINE

1. Parametrization and Problem Formulation
2. Lean Buffering in Synchronous Lines with Identical Exponential machines
3. Lean Buffering in Synchronous Lines with Non-Identical Exponential Machines
4. Lean Buffering in Synchronous Lines with Non-Exponential Machines
5. Summary

1. Parametrization and Problem Formulation

- Assume, for the moment, that

$$T_{up,i} =: T_{up}, \qquad T_{down,i} =: T_{down}, \qquad i = 1, \ldots, M,$$
$$N_i =: N, \qquad i = 1, \ldots, M-1.$$

- Parametrizations:
  - Level of buffering – capacity of the buffer in units of downtime:

$$k := \frac{N}{T_{down}}.$$

  - Line efficiency:

$$E := \frac{PR}{PR_\infty}.$$

  - **Definition**: Lean level of buffering (k_E) – the smallest level of buffering necessary and sufficient to ensure E.

Problems

- Develop methods for calculating k_E for exponential lines.
- Investigate sensitivity of k_E to type of up- and downtime distributions in non-exponential lines.
- Based on this analysis, provide empirical formulas for calculating k_E for non-exponential lines.
- Generalize the results to systems with non-identical machines and buffers.

2. Lean Buffering in Synchronous Lines with Identical Exponential machines

2.1 Two-machine case

- In this case:

$$\lambda_1 = \lambda_2 =: \lambda, \qquad \mu_1 = \mu_2 =: \mu,$$

$$Q(\lambda_1, \mu_1, \lambda_2, \mu_2, N) = Q(\lambda, \mu, N) = \frac{2\lambda}{(\lambda + \mu)[2 + (\lambda + \mu)N]}.$$

$$PR = Ee = e[1 - Q(\lambda, \mu, N_E)].$$

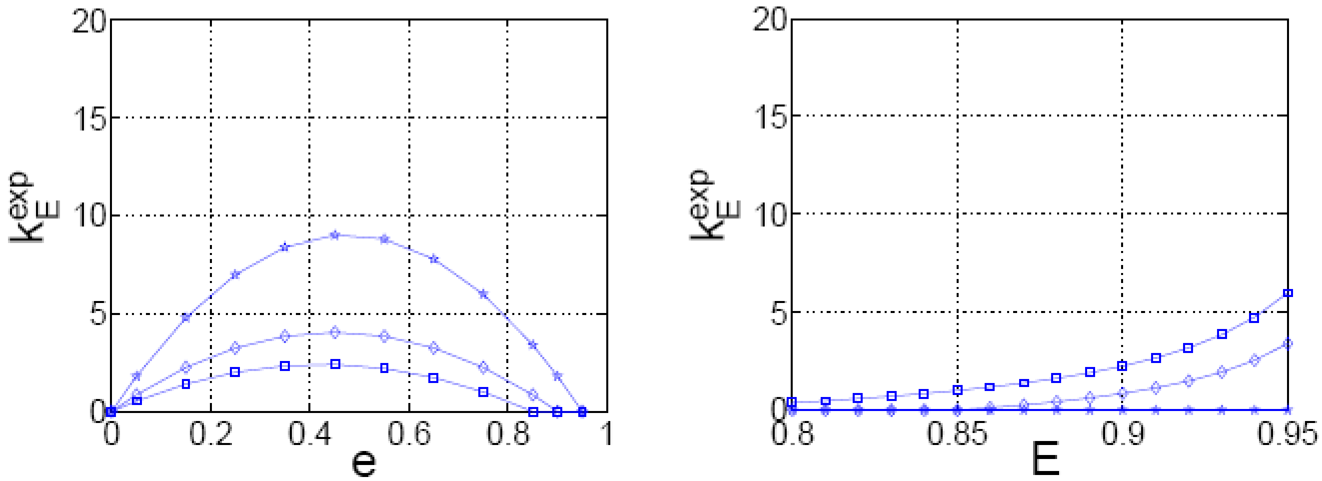
⇓

$$N_E = \begin{cases} \frac{2(1-e)(E-e)}{\lambda(1-E)}, & \text{if } e < E, \\ 0, & \text{otherwise.} \end{cases}$$

- **Theorem:**

$$k_E^{exp}(M = 2) = \begin{cases} \frac{2e(E-e)}{1-E}, & if\ e < E, \\ 0, & otherwise. \end{cases}$$

2.1 Two-machine case (cont)



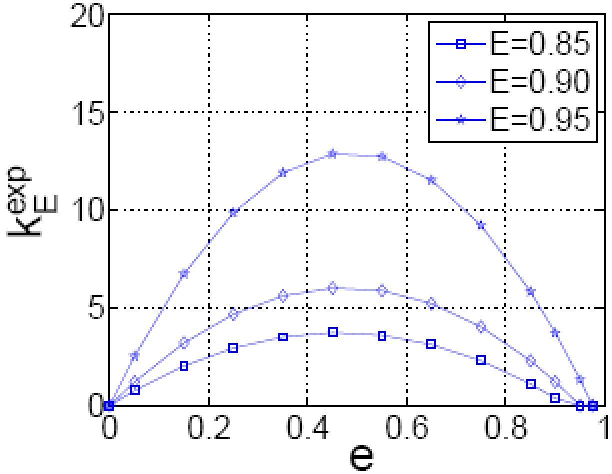
- Observations:
  - $k_E^{exp}(M=2)$ does not depend on T_{up} and T_{down} (only T_{up}/T_{down}).
  - If $E > e$, the system must have a buffer.
  - If $E \le e$, just-in-time is possible.
  - $k_E^{exp}(M=2) \to \infty$ as $E \to 1$.

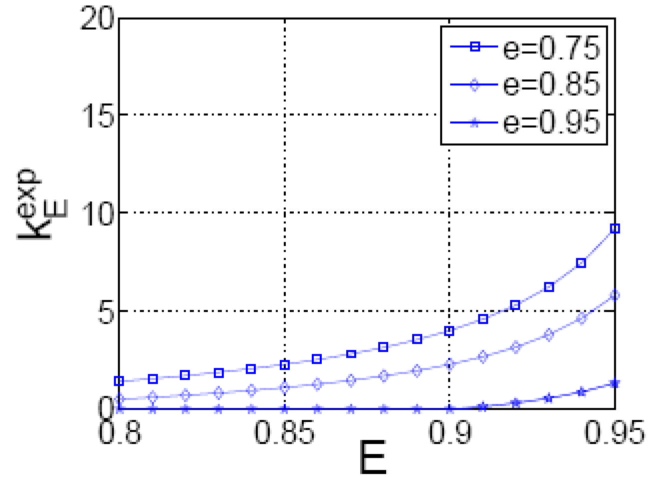
2.2 Three-machine case

- **Theorem:**

$$k_E^{exp}(M=3) = \begin{cases} \frac{e(1+\sqrt{E})(e+e\sqrt{E}-2)}{2(1-\sqrt{E})} \ln\left(\frac{1-e\sqrt{E}}{(1-e)(1+\sqrt{E})}\right), & if\ e < \sqrt{E}, \\ 0, & otherwise. \end{cases}$$

2.2 Three-machine case (cont)





- Observations:
  - Again, $k_E^{exp}(M=3)$ does not depend on T_{up} and T_{down}.
  - If $\sqrt{E} > e$, the system must have a buffer.
  - If $\sqrt{E} > e$, just-in-time is possible.
  - $k_E^{exp}(M=3) \rightarrow \infty$ as $E \rightarrow 1$.

2.3 *M* > 3-machine case

- **Theorem:**

$$k_E^{exp}(M>3)=\begin{cases}\frac{e(2-Q)(2e-eQ-2)}{2Q}\ln\left(\frac{E-eE+eEQ-1+e-2eQ+eQ^2+Q}{(1-e-Q+eQ)(E-1)}\right), & if\; e<E^{\frac{1}{M-1}},\\ 0, & otherwise,\end{cases}$$

where

$$Q=Q(\lambda_{M-2}^{f},\mu_{M-2}^{f},\lambda_{M-1}^{b},\mu_{M-1}^{b},N_E)$$

- Approximation of Q:

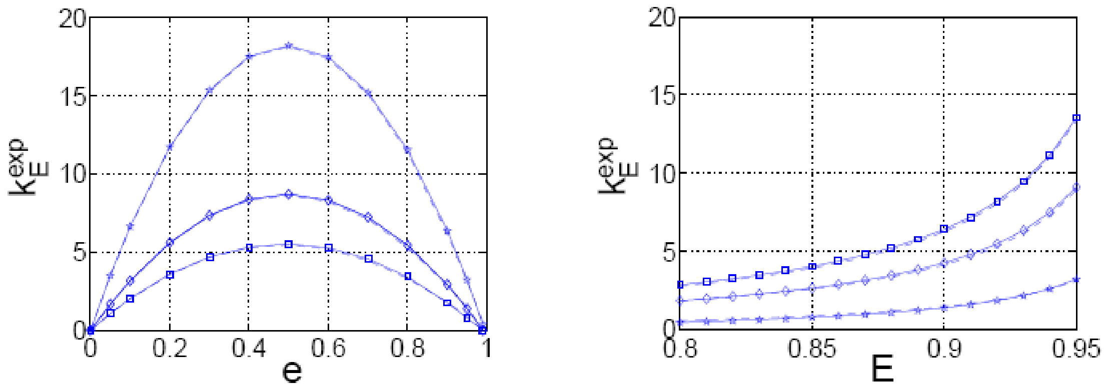
$$\widehat{Q}=1-E^{\frac{1}{2}\left[1+\left(\frac{M-3}{M-1}\right)^{M/4}\right]}+\left(E^{\frac{1}{2}\left[1+\left(\frac{M-3}{M-1}\right)^{M/4}\right]}-E^{\frac{M-2}{M-1}}\right)\exp\left\{-\left(\frac{E^{\frac{1}{M-1}}-e}{1-\sqrt{E}}\right)\right\}.$$

- Accuracy: $\Delta_Q=\frac{Q-\widehat{Q}}{Q}\cdot 100\%,\;\Delta_{k_E}=\frac{\widehat{k}_E-k_E}{k_E}\cdot 100\%$

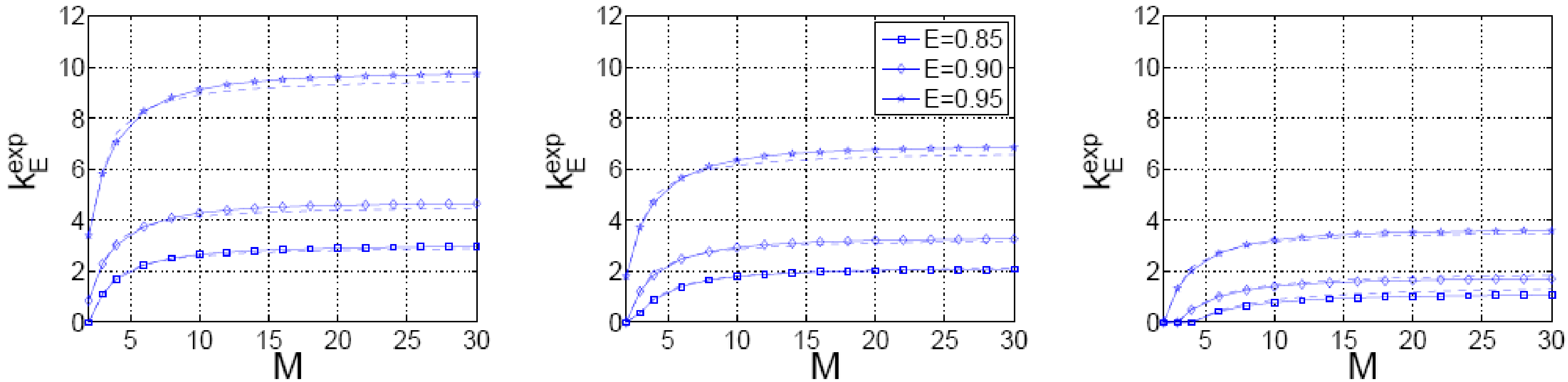
$$\Delta_Q,\;\Delta_{k_E}\leq 5\%.$$

2.3 *M* > 3-machine case (cont)

- Behavior of k_E as a function of e



- Behavior of k_E as a function of M



(a) $e = 0.85$ (b) $e = 0.9$ (c) $e = 0.95$

Rule-of-thumb: For $M \geq 10$

e	$E = 0.85$	$E = 0.90$	$E = 0.95$
0.85	3.4	5	9.8
0.90	2.7	3.9	7.2
0.95	1.6	2.4	4.3

- Q: Does buffer $N = 1000$ lean or not?
- A: It depends: If $T_{down} = 1000$, the buffer is too lean (since $k_E = 1$). If $T_{down} = 10$, the buffer is very much not lean (since $k_E = 100$).

3. Lean Buffering in Synchronous Lines with Non-Identical Exponential Machines

3.1 Two-machine case

- Parametrization

$$k_i = \frac{N_i}{\max(T_{down,i}, T_{down,i+1})}$$

- $PR = E \cdot \min(e_1, e_2) = e_2[1 - Q(\lambda_1, \mu_1, \lambda_2, \mu_2, N_E)]$
- **Theorem:**

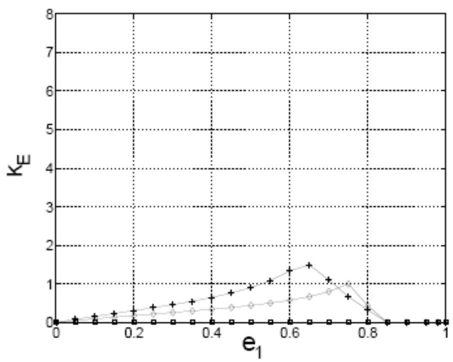
$$k_E^{exp}(M=2) = \begin{cases} \frac{\min(\mu_1,\mu_2)}{\beta} \ln \left\{ \phi \frac{(e_2 - E \cdot PR_\infty)}{(e_1 - E \cdot PR_\infty)} \right\}, & \textit{if } \min(e_1, e_2) < E, \\ 0 & \textit{otherwise.} \end{cases}$$

$$\phi = \frac{e_1(1-e_2)}{e_2(1-e_1)},$$

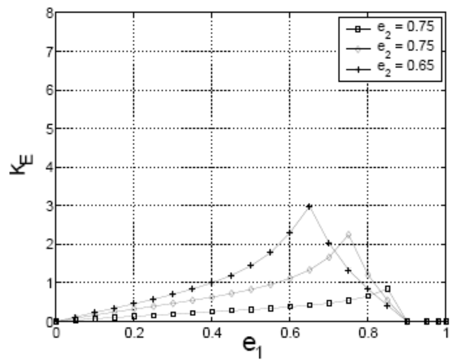
$$\beta = \frac{(\mu_1 + \mu_2 + \lambda_1 + \lambda_2)(\lambda_1\mu_2 - \lambda_2\mu_1)}{(\mu_1 + \mu_2)(\lambda_1 + \lambda_2)}.$$

3.1 Two-machine case (cont)

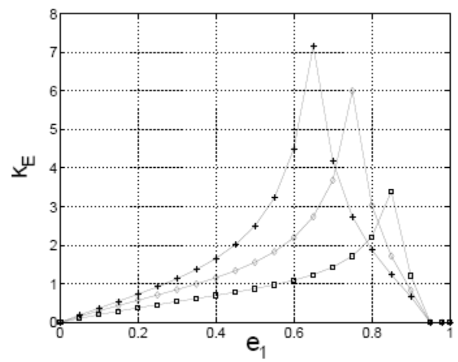
- Behavior of k_E as functions of e and E



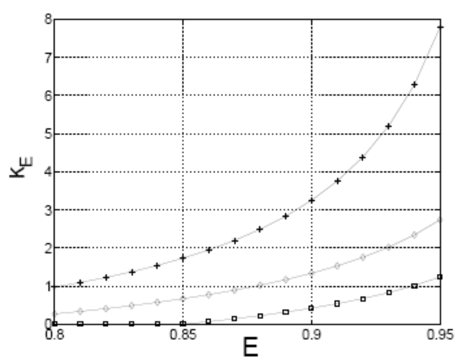
(a) $E = 0.85$



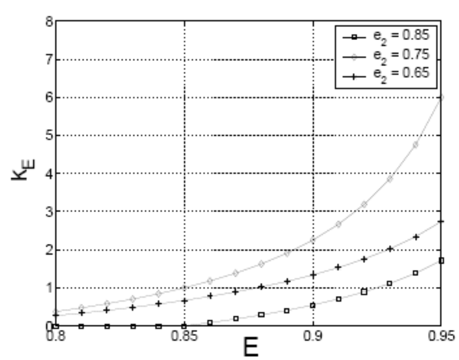
(b) $E = 0.90$



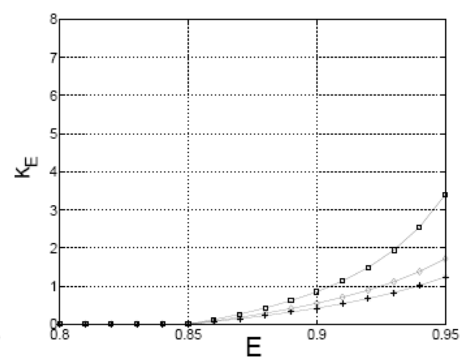
(c) $E = 0.95$



(a) $e_1 = 0.65$



(b) $e_1 = 0.75$



(c) $e_1 = 0.85$

3.1 Two-machine case (cont)

- Observations:
  - $k_E^{exp}(M=2)$ depends on T_{up} and T_{down} explicitly.
  - If $E > \min(e_1, e_2)$, the systems must have buffers.
  - If $E \leq \min(e_1, e_2)$, JIT is possible.

3.2 *M* > 2-machine case

- *Closed formula* and *recursive* approaches to estimate $k_{i,E}$, $i = 1,\ldots, M-1$, are developed (similar to those for Bernoulli lines)
- Closed formula approaches:
  I. Local pair-wise approach: use the two-nonidentical-machine formula for consecutive machines:

$$k^I_{i,E},\ i = 1, \ldots, M-1$$

  II. Global pair-wise approach: Use the two-nonidentical-machine formula for possible pairs of machines:

$$k^{II}_E$$

Closed formula approaches

III. <u>Local upper bound approach</u>: Substitute each pair of consecutive machines by

$$\widehat{e}_i = \min(e_i, e_{i+1}), \quad \widehat{\mu}_i = \min(\mu_i, \mu_{i+1}), \quad \widehat{\lambda}_i = \widehat{\mu}_i(1-\widehat{e}_i)/\widehat{e}_i$$

and use the two-identical-machines formula:

$$k_{1,E}^{III}, \ldots, k_{M-1,E}^{III}$$

IV. <u>Global upper bound approach</u>: Substitute the line by an *M*-machine line with identical machines defined by

$$\widehat{e} = \min(e_1, \ldots, e_M), \; \widehat{\mu} = \min(\mu_1, \ldots, \mu_M), \; \widehat{\lambda} = \widehat{\mu}(1-\widehat{e})/\widehat{e}$$

and use *M*-identical-machine formula:

$$k_E^{IV}$$

Measures of quality of each approach

- Average level of buffering

$$k^j_{ave} = \frac{1}{S}\sum_{s=1}^{S} k^j_s, \quad S = 100,000$$

$$\begin{aligned} M_s &\in \{5, 10, 15, 20, 25, 30\}, \\ e_{i,s} &\in [0.7, 0.97], \\ T_{down,i,s} &\in \{5, 10, \ldots, 50\}, \\ E_s &\in [0.8, 0.98]. \end{aligned}$$

where

$$k^j_s = \frac{1}{M_s - 1}\sum_{i=1}^{M_s - 1} k^j_{i,s}.$$

- Fraction of systems where E_s is not reached:

$$\Delta^j = \frac{1}{S}\sum_{s=1}^{S} \mathrm{Sg}\{E_s - E^j_s\} \cdot 100\%$$

$$\mathrm{Sg}(x) = \begin{cases} 1, & \text{when } x > 0, \\ 0, & \text{when } x \le 0. \end{cases}$$

Results

Approach	I	II	III	IV
k^j_{ave}	1.2	5.0	5.4	9.5
Δ^j	96.7	0.9	0.0	0.0

Observation:

- Local pair-wise selection of k_E is not acceptable.
- Global pair-wise and local upper bound approaches are the best.

Effect of *M* on approaches I-IV

M	5	10	15	20	25	30	50	80
k_{ave}^{I}	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2
k_{ave}^{II}	3.2	4.5	5.1	5.5	5.7	5.9	6.4	6.6
k_{ave}^{III}	5.3	5.5	5.5	5.4	5.4	5.5	5.4	5.4
k_{ave}^{IV}	7.8	9.3	9.8	10.0	10.0	10.3	10.4	10.4
Δ^{I}	88.4	96.2	98.2	99.0	99.4	99.5	99.9	99.9
Δ^{II}	0.6	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Δ^{III}	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Δ^{IV}	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Observation:

- Global pair-wise approach is the best for $M \leq 15$.
- For $M > 15$, local upper bound approach is preferable.

Recursive approaches:

V. Full search approach:

$$k_{1,E}^{V}, \ldots, k_{M-1,E}^{V}$$

VI. Bottleneck-based approach: Use Approach I, identify *c*-BN and increase buffers around this machine:

$$k_{1,E}^{VI}, \ldots, k_{M-1,E}^{VI}$$

- Performance metrics:

  As above + time ($t^j = t_{end}^j - t_{start}^j$)

Results

(a) $E = 0.80$

Approaches	I	II	III	IV	V	VI
k^j_{ave}	0.4	1.1	1.6	2.7	0.5	0.6
Δ^j	92.4	2.3	0.0	0.0	0.0	0.0
t^j	0	0	0	0	132	8

(b) $E = 0.85$

Approaches	I	II	III	IV	V	VI
k^j_{ave}	0.6	1.7	2.4	3.8	0.7	0.9
Δ^j	93.0	0.8	0.0	0.0	0.0	0.0
t^j	0	0	0	0	211	12

(c) $E = 0.90$

Approaches	I	II	III	IV	V	VI
k^j_{ave}	1.0	2.6	4.0	6.0	1.1	1.4
Δ^j	90.2	0.2	0.0	0.0	0.0	0.0
t^j	0	0	0	0	355	17

(d) $E = 0.95$

Approaches	I	II	III	IV	V	VI
k^j_{ave}	2.0	5.0	8.6	12.6	1.9	2.4
Δ^j	81.2	0.0	0.0	0.0	0.0	0.0
t^j	0	0	0	0	679	22

Observations

- Full search results in smallest buffering but longest time.
- Approaches II-IV are practically instantaneous but lead to 2-10 larger buffering than approach V.
- Approach VI provides a good trade off: 20% larger buffering than V but up to two order of magnitude faster.

Illustration

Line	E	e_1	e_2	e_3	e_4	e_5
1	0.80	0.83	0.88	0.71	0.74	0.90
2	0.85	0.97	0.76	0.79	0.75	0.90
3	0.90	0.91	0.87	0.76	0.84	0.78
4	0.95	0.79	0.88	0.96	0.95	0.81

(a). Line efficiencies

Line	E	$T_{down,1}$	$T_{down,2}$	$T_{down,3}$	$T_{down,4}$	$T_{down,5}$
1	0.80	22	39	17	23	28
2	0.85	22	24	49	47	30
3	0.90	33	20	31	27	29
4	0.95	32	15	35	37	19

(b). Machine parameters

Illustration (cont)

Buffer	b_1^j	b_2^j	b_3^j	b_4^j	E^j
Desired					0.80
k_i^I	0.3	0.2	1.1	0.1	0.71
k_i^{II}	1.9	1.9	1.1	1.4	0.87
k_i^{III}	1.2	2.9	1.7	1.8	0.90
k_i^{IV}	2.9	2.9	2.9	2.9	0.95
k_i^V	0.5	1.0	1.4	0.3	0.80
k_i^{VI}	0.3	1.1	1.9	0.1	0.80

LLB estimates for Line 1

Buffer	b_1^j	b_2^j	b_3^j	b_4^j	E^j
Desired					0.85
k_i^I	0.0	1.7	2.1	0.2	0.81
k_i^{II}	1.2	2.6	2.6	2.4	0.89
k_i^{III}	1.8	3.7	3.8	3.6	0.93
k_i^{IV}	3.8	3.8	3.8	3.8	0.93
k_i^V	0.2	2.1	2.3	0.8	0.85
k_i^{VI}	0.0	2.0	2.8	0.7	0.85

LLB estimates for Line 2

Illustration (cont)

Buffer	b_1^j	b_2^j	b_3^j	b_4^j	E^j
$Desired$					0.90
k_i^I	0.8	1.0	1.8	1.9	0.84
k_i^{II}	4.4	4.1	4.1	3.8	0.95
k_i^{III}	2.7	5.4	5.4	4.8	0.97
k_i^{IV}	5.9	5.9	5.9	5.9	0.97
k_i^V	0.7	2.2	3.0	2.5	0.90
k_i^{VI}	0.8	2.4	3.3	1.9	0.90

LLB estimates for Line 3

Buffer	b_1^j	b_2^j	b_3^j	b_4^j	E^j
$Desired$					0.95
k_i^I	1.6	0.4	0.7	0.6	0.82
k_i^{II}	5.5	6.1	6.4	6.4	1.00
k_i^{III}	9.4	6.1	1.5	10.3	1.00
k_i^{IV}	10.9	10.9	10.9	10.9	1.00
k_i^V	3.5	2.8	1.5	3.2	0.95
k_i^{VI}	2.9	1.8	0.8	6.4	0.95

LLB estimates for Line 4

4. Lean Buffering in Synchronous Lines with Non-exponential Machines

4.1 Approach

- Select a representative set of up- and downtime distributions.
- By simulations, evaluate PR and k_E.
- Investigate the sensitivity of k_E to $f_{t_{up}}$ and $f_{t_{down}}$
- Derive an empirical formula for k_E.

Systems analyzed

- Identical reliability model

$$\{[W(\lambda, \Lambda), W(\mu, \mathrm{M})]_i, \qquad i = 1, \ldots, 10\}$$
$$\{[ga(\lambda, \Lambda), ga(\mu, \mathrm{M})]_i, \qquad i = 1, \ldots, 10\}$$
$$\{[LN(\lambda, \Lambda), LN(\mu, \mathrm{M})]_i, \qquad i = 1, \ldots, 10\}$$

- Mixed reliability model

Line 1: $\{(ga, W), (LN, LN), (W, ga), (ga, LN), (ga, W)$
$(LN, ga), (W, W), (ga, ga), (LN, W), (ga, LN)\}$

Line 2: $\{(W, LN), (ga, W), (LN, W), (W, ga), (ga, LN)$
$(ga, W), (W, W), (LN, ga), (ga, W), (LN, LN)\}$

Systems analyzed (cont)

- Assumptions

$$CV_{up,i} = CV_{down,i} =: CV, \quad i = 1, \ldots, M,$$

or

$$CV_{up,i} =: CV_{up}, \quad CV_{down,i} =: CV_{down}, \quad i = 1, \ldots, M$$

and

$$T_{up,i} =: T_{up}, \quad T_{down,i} =: T_{down}, \quad i = 1, \ldots, M.$$

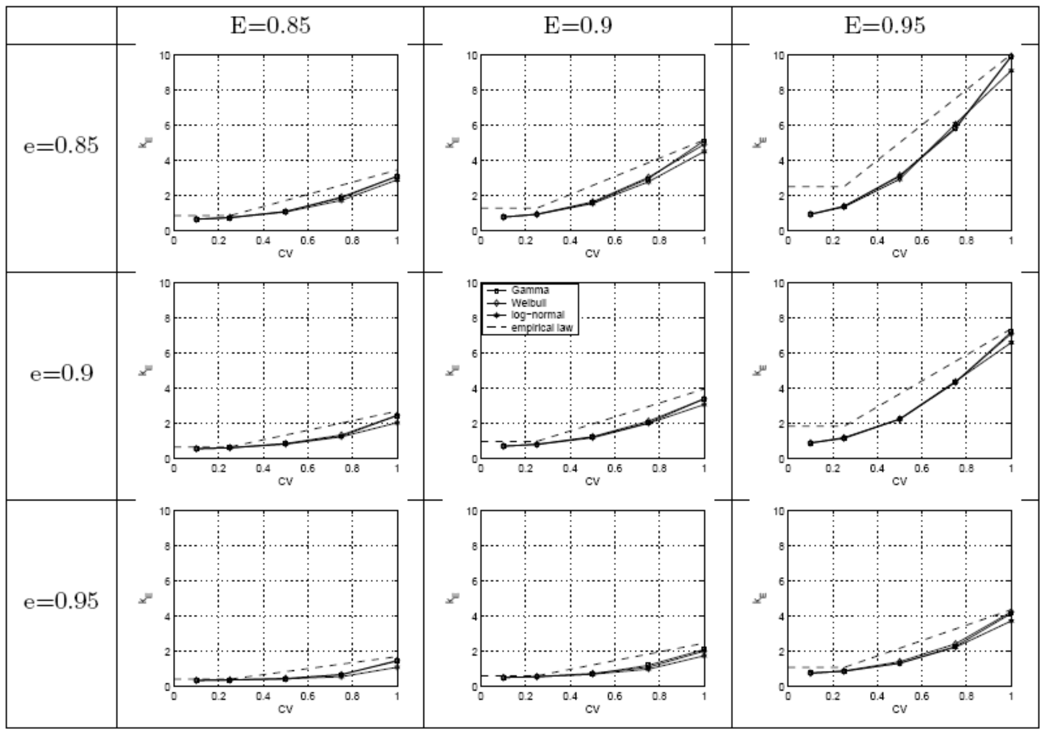
Reliability models considered

CV_{down}	$T_{down} = 10$
0.1	$W(0.096, 12.15)$, $ga(10, 100)$, $LN(2.30, 0.1)$
0.25	$W(0.092, 4.54)$, $ga(1.6, 16)$, $LN(2.27, 0.25)$
0.5	$W(0.088, 2.1)$, $ga(0.4, 4)$, $LN(2.19, 0.47)$
0.75	$W(0.092, 1.35)$, $ga(0.18, 1.78)$, $LN(2.08, 0.67)$
1.00	$LN(1.96, 0.83)$

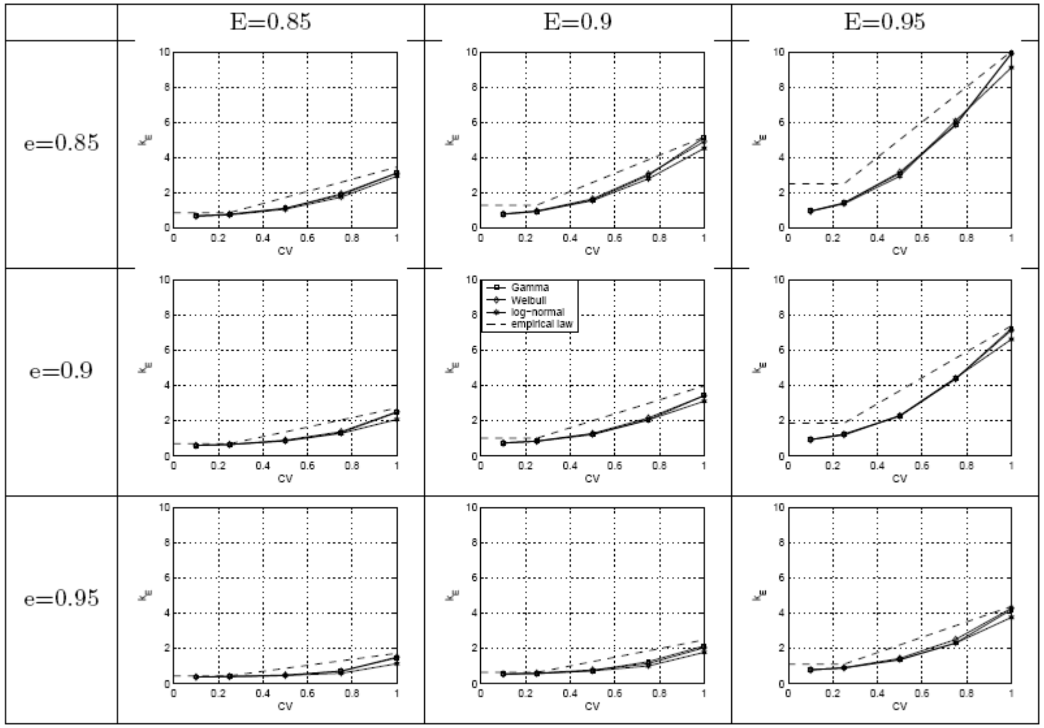
CV_{down}	$T_{down} = 20$
0.1	$W(0.048, 12.15)$, $ga(5, 100)$, $LN(2.99, 0.1)$
0.25	$W(0.046, 4.54)$, $g(0.8, 16)$, $LN(2.97, 0.25)$
0.5	$W(0.044, 2.1)$, $ga(0.2, 4)$, $LN(2.88, 0.47)$
0.75	$W(0.046, 1.35)$, $ga(0.09, 1.78)$, $LN(2.77, 0.67)$
1.00	$LN(2.65, 0.83)$

4.2 Sensitivity of k_E to machine reliability model

- Case $CV_{up} = CV_{down} =: CV$



■ Case $CV_{up} = CV_{down} =: CV$



Observations:

- k_E is monotonically increasing as a function of CV.
- k_E for various reliability models are almost the same, i.e., k_E is not sensitive to $f_{t_{up}}$ and $f_{t_{down}}$ (at most 10% difference)

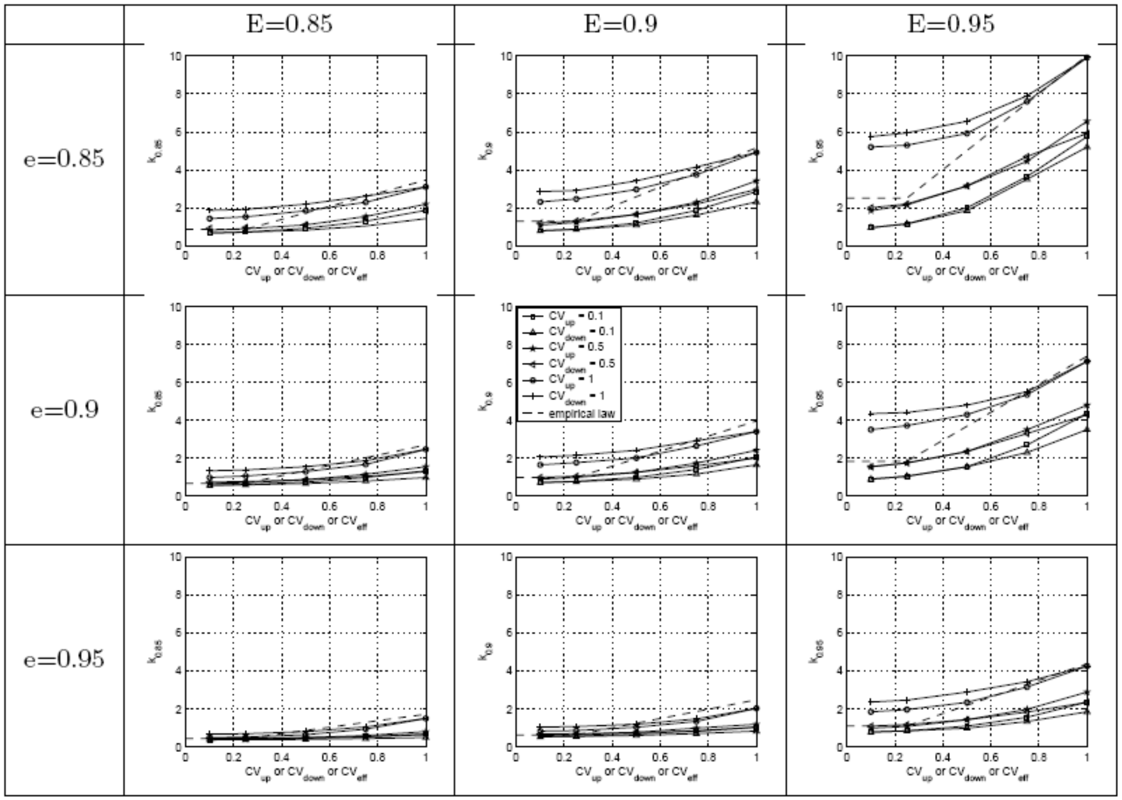
Case $CV_{up} \neq CV_{down}$

- What affects k_E more: CV_{up} or CV_{down}?
- Formalization of this question: Inequality

$$k_E(CV_{down} = CV | CV_{up} = \alpha) < k_E(CV_{up} = CV | CV_{down} = \alpha)$$

  implies that CV_{down} has a larger effect.
- If the inequality is reversed, CV_{up} has a larger effect.

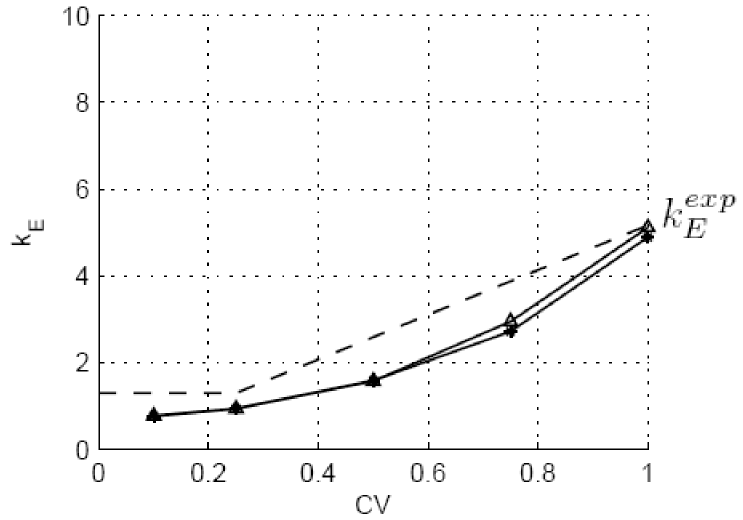
Results



- Thus, CV_{down} has a larger effect on k_E than CV_{up}.
- However, the difference is not too dramatic (within 30%).

4.3 Empirical bound for $k_E(M, E, e, CV)$

- Case $CV_{up} = CV_{down} =: CV$

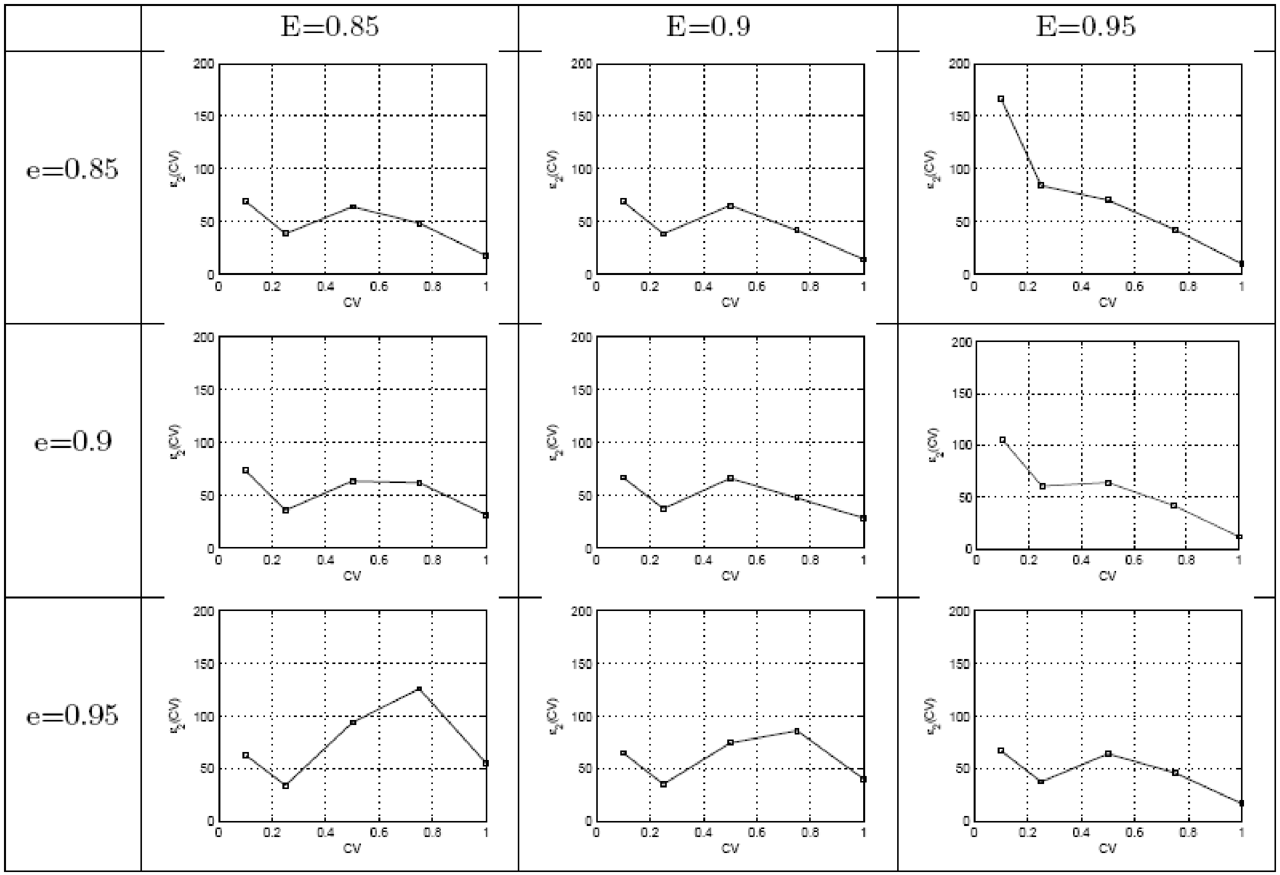


$$k_E(M,E,e,CV) \leq \begin{cases} k_E^{exp}(M,E,e)\cdot CV, & \text{if } 0.25 < CV \leq 1, \\ 0.25k_E^{exp}(M,E,e), & \text{if } \quad 0 \leq CV \leq 0.25, \end{cases}$$

or

$$k_E(M,E,e,CV) \leq \max\{0.25, CV\}k_E^{exp}(M,E,e).$$

▪ Tightness of the empirical bound



Case $CV_{up} \neq CV_{down}$

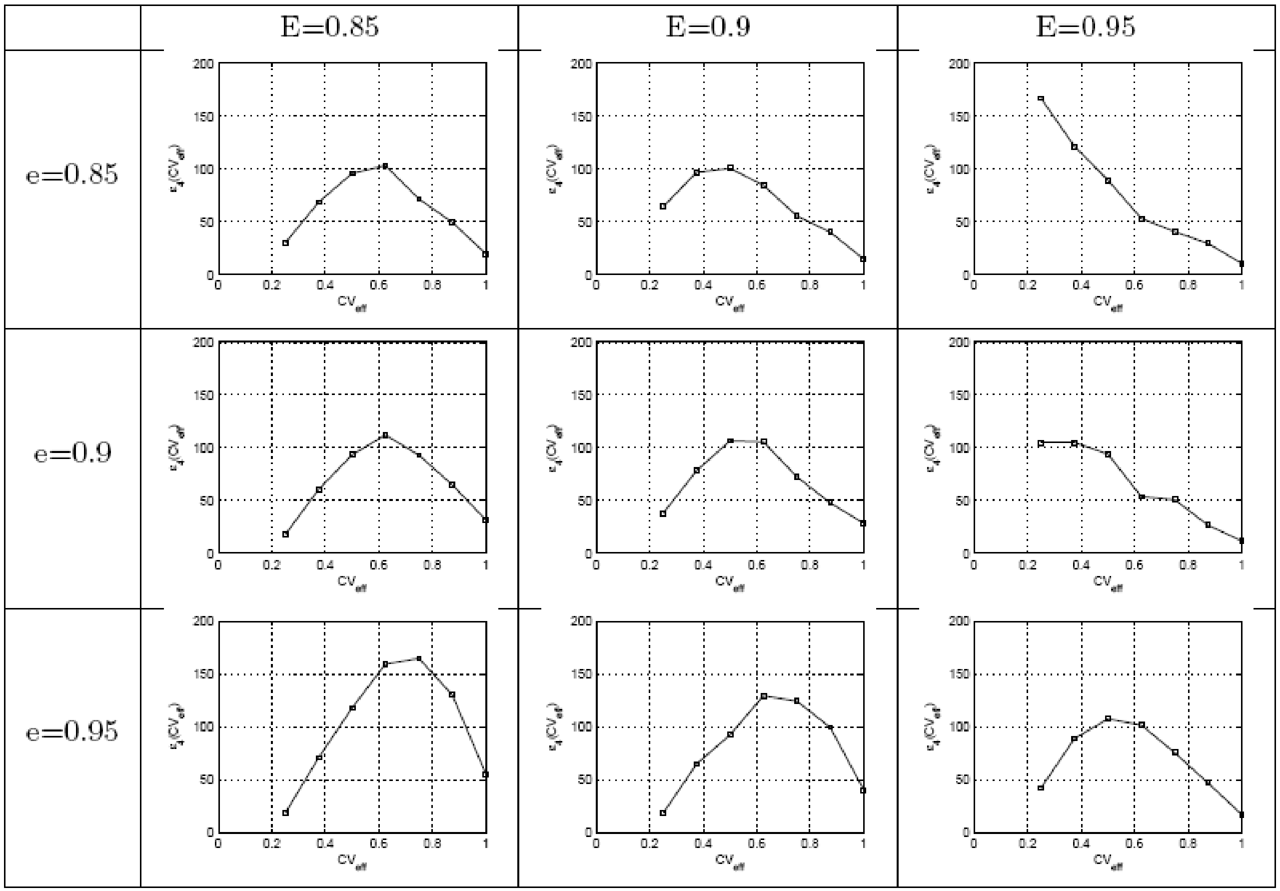
- Define

$$CV_{eff} := \frac{\max\{0.25, CV_{up}\} + \max\{0.25, CV_{down}\}}{2}.$$

- Then

$$k_E \leq k_E^{exp}(M, E, e) \cdot CV_{eff}.$$

Tightness



Generalization to non-identical machines

- To account for $T_{up,\,i} \neq T_{up,\,l}$, $T_{down,\,i} \neq T_{down,\,l}$, use

$$k^j_{i,E},\ i = 1,\ldots,M-1,\ j = I,\ldots,VI.$$

- To account for $CV_{up,\,i} \neq CV_{up,\,l}$, $CV_{down,\,i} \neq CV_{down,\,l}$, use

$$CV_{eff,i} := \frac{\max\{0.25, CV_{up,i}\} + \max\{0.25, CV_{down,i}\}}{2}, \quad i = 1,\ldots,M.$$

- Then

$$k^{j,non-exp}_{i,E} \leq \frac{CV_{eff,i} + CV_{eff,i+1}}{2} \cdot k^j_{i,E}, \quad i = 1,\ldots,M-1,$$
$$j \in \{I,\ldots,VI\}$$

5. Summary

- Lean buffering in serial lines with continuous time models of machine reliability can be characterized in terms of two dimensionless parameters: the level of buffering (which quantifies buffer capacities in units of downtime) and the line efficiency (which quantifies the desired production rate in units of the largest production rate available in the system).
- Closed formulas for the lean level of buffering (LLB) have been derived for exponential lines with identical machines.
- For exponential lines with non-identical machines, closed formulas are available only for two- and three-machine cases. For longer lines, LLB approximations, based on either closed formulas or recursions, have been developed.

- Among the closed formulas approaches, the global pair-wise approach is recommended in lines with less than 15 machines; otherwise, the local upper bound approach is deemed the best. Among the recursive approaches, the bottleneck-based one is preferred.
- For lines with non-exponential machines, it is shown that LLB is not sensitive to the shape of up- and downtime distribution and depends mainly on their coefficients of variation.
- Based on this observation, empirical formulas are derived, which provide an upper bound on LLB as a function of up- and downtime *CV*'s, given that both of them are less than 1.